L^q BOUNDS ON RESTRICTIONS OF SPECTRAL CLUSTERS TO SUBMANIFOLDS FOR LOW REGULARITY METRICS

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Abstract. We prove L^q bounds on the restriction of spectral clusters to submanifolds in Riemannian manifolds equipped with metrics of $C^{1,\alpha}$ regularity for $0 \le \alpha \le 1$. Our results allow for Lipschitz regularity when $\alpha = 0$, meaning they give estimates on manifolds with boundary. When $0 < \alpha \le 1$, the scalar second fundamental form for a codimension 1 submanifold can be defined, and we show improved estimates when this form is negative definite. This extends results of Burq-Gérard-Tzvetkov and Hu to manifolds with low regularity metrics.

1. Introduction

Let M be a compact, smooth manifold of dimension $n \ge 2$ equipped with Riemannian metric g of at least Lipschitz regularity. Let Δ_{g} denote the associated (negative) Laplace-Beltrami operator whose action in coordinates is given by the differential operator

$$\Delta_{\mathrm{g}} f = \frac{1}{\sqrt{\det \mathrm{g}_{kl}}} \sum_{i,j} \partial_i \left(\mathrm{g}^{ij} \sqrt{\det \mathrm{g}_{kl}}\, \partial_j f \right).$$

There exists an orthonormal basis $\{\phi_j\}_{j=1}^\infty$ of $L^2(M)$ consisting of eigenfunctions of Δ_{g}, which can be seen by passing to quadratic forms (see e.g. [19, §1]). We write the corresponding Helmholtz equation for ϕ_j as $(\Delta_{\mathrm{g}} + \lambda_j^2)\phi_j = 0$ so that λ_j gives the frequency of vibration associated to ϕ_j.

Given $\lambda \ge 1$, we let Π_λ be the projection operator on $L^2(M)$ defined by $\Pi_\lambda f := \sum_{\lambda_j \in [\lambda, \lambda+1]} \langle f, \phi_j \rangle \phi_j$, where $\langle \cdot, \cdot \rangle$ denotes the usual L^2 inner product with respect to the Riemannian measure. We call functions f which are in the range of some Π_λ "spectral clusters". They form approximate eigenfunctions or quasimodes as $\|(\Delta_{\mathrm{g}} + \lambda^2)\Pi_\lambda f\|_{L^2(M)} \le C\lambda \|f\|_{L^2(M)}$. In [22], Sogge proved that when g is a C^∞ metric, the following L^q bounds on the projections $\Pi_\lambda f$ are satisfied for $q \ge 2$

$$\|\Pi_\lambda f\|_{L^q(M)} \le C\lambda^\delta \|f\|_{L^2(M)}, \tag{1.1}$$

where $\delta = \delta(q) = \max(\frac{n-1}{2}(\frac12 - \frac1q), n(\frac12 - \frac1q) - \frac12)$. He also provided examples showing that the exponent $\delta(q)$ is the best possible for these approximate eigenfunctions. Since Π_λ is a projection operator, any L^q bound it satisfies implies L^q bounds on individual eigenfunctions. Determining when these bounds are sharp for subsequences of eigenfunctions is an area of active interest, though we do not examine this issue here.

In [18], Smith proved that the bounds (1.1) are satisfied for $C^{1,1}$ metrics. The assumption of $C^{1,1}$ regularity is the lowest degree of continuity needed to ensure the

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uniqueness of geodesics on M. Since eigenfunctions naturally give rise to solutions to the wave equation, propagation of singularities suggests that this is a relevant consideration for the validity of such bounds. Indeed, works of Smith-Sogge [20] and Smith-Tataru [21] give examples of $C^{1,\alpha}$ metrics (Lipschitz when $\alpha = 0$) which give rise to spectral clusters $\Pi_\lambda f_\lambda = f_\lambda$ for each $\lambda \geq 1$ such that

$$\frac{\|f_\lambda\|_{L^q(M)}}{\|f_\lambda\|_{L^2(M)}} \geq c\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})(1+\sigma)}, \qquad \sigma = \frac{1-\alpha}{3+\alpha}, \tag{1.2}$$

showing that the bounds (1.1) cannot hold for $2 < q < \frac{2(n+2(1+\alpha)^{-1})}{n-1}$. In each case, the cluster f_λ is highly concentrated in a tube about a curve segment of length 1 and diameter $\lambda^{-\frac{2}{3+\alpha}}$ (cf. (1.10) below). This shows that the family $\{f_\lambda\}_{\lambda\geq 1}$ exhibits a greater degree of concentration than Sogge's examples which saturate the bounds (1.1) when $2 < q \leq \frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})$ (they are concentrated in tubes with with diameter $\lambda^{-\frac{1}{2}}$). In [19], Smith showed positive results for any $C^{1,\alpha}$ metric, proving that the that the ratio on the left in (1.2) is always bounded above by $C\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})(1+\sigma)}$ when $2 \leq q \leq \frac{2(n+1)}{n-1}$. He also proved that the bound (1.1) holds when $q = \infty$. By interpolation, this shows (1.1) with a loss of σ/q derivatives when $\frac{2(n+1)}{n-1} \leq q \leq \infty$.

In a similar vein, when $\mathrm{g} \in C^\infty$, results of Burq-Gérard-Tzvetkov [4], Hu [11], and Reznikov [15] show L^q bounds on the restriction of these spectral clusters to embedded submanifolds $P \subset M$ of the form

$$\|\Pi_\lambda f\|_{L^q(P)} \leq C\lambda^\delta \|f\|_{L^2(M)} \qquad q \geq 2 \tag{1.3}$$

where $\|\Pi_\lambda f\|_{L^q(P)}$ taken to mean the L^q norm of the restriction $\Pi_\lambda f\big|_P$. In this case, $\delta = \delta(k,q)$ depends on the dimension of the submanifold k and on q. In particular, when $k = n-1$, $\delta = \max(\frac{n-1}{2} - \frac{n-1}{q}, \frac{n-1}{4} - \frac{n-2}{2q})$, that is,

$$\delta(n-1,q) = \begin{cases} \frac{n-1}{2} - \frac{n-1}{q}, & \text{if } \frac{2n}{n-1} \leq q \leq \infty, \\ \frac{n-1}{4} - \frac{n-2}{2q}, & \text{if } 2 \leq q \leq \frac{2n}{n-1} \end{cases}. \tag{1.4}$$

Otherwise, when $1 \leq k \leq n-2$,

$$\delta(k,q) = \frac{n-1}{2} - \frac{k}{q} \tag{1.5}$$

with the exception of $(k,q) = (n-2,2)$ where there is a logarithmic loss for $\lambda \geq 2$, $\|\Pi_\lambda f\|_{L^2(P)} \leq C(\log\lambda)^{\frac{1}{2}}\lambda^{\frac{1}{2}}\|f\|_{L^2(M)}$. These bounds were proved in a semiclassical setting by Tacy [24]. We also remark that the bound (1.3) in the case $k = n-1$, $q = 2$ was previously observed by Tataru [25] as a consequence of the estimates in Greenleaf-Seeger [8]. As will be discussed in §2, these bounds provide an improvement over what would be obtained by trace theorems for Sobolev spaces.

One reason the bounds (1.1), (1.3) are of such great interest is that they illuminate the size and concentration properties of eigenfunctions. In particular, Smith's work on $C^{1,\alpha}$ metrics [19] is significant in that it addresses concentration phenomena in situations where the roughness of the metric means that geodesic curves may fail to be unique. It also led to the development of sharp bounds of the form (1.1) for the Dirichlet and Neumann Laplacians on compact Riemannian manifolds with boundary (see [17]). Indeed, one strategy for proving estimates in this context is to form the double of the manifold, essentially gluing two copies of the manifold along

the boundary. While this eliminates the boundary, it gives rise to a metric of Lipschitz regularity (see e.g. [3, p.420]). Hence any result on manifolds with Lipschitz metrics also applies to manifolds with boundary. Moreover, the bounds (1.3) when $n = 2$, $k = 1$ (curves in 2 dimensional manifolds) have garnered additional interest in recent works which relate improvements in these estimates to improvements in the inequalities in (1.1) (see [2], [23], [1]).

On the other hand, one of the notable aspects of the work of Burq-Gérard-Tzvetkov [4] is that they showed an improvement on (1.3) when $n = 2$ and P is a curve with nonvanishing geodesic curvature. Specifically, they proved that

$$\|\Pi_\lambda f\|_{L^2(P)} \le C\lambda^{\frac{1}{6}}\|f\|_{L^2(M)}. \tag{1.6}$$

This was then generalized to all dimensions by Hu [11] who obtained the same bound for any codimension 1 submanifold with negative definite scalar second fundamental form (or positive definite, depending on the choice of normal vector). As before, these bounds also follow from an observation of Tataru [25] based on known estimates of Hörmander [10, 25.3]. The bound (1.6) can then be interpolated with (1.3) when $q = \frac{2n}{n-1}$, $\delta = \frac{n-1}{2n}$ to get that the δ in (1.4) can be improved to

$$\delta = \frac{n-1}{3} - \frac{2n-3}{3q} \qquad \text{when } 2 \le q < \frac{2n}{n-1}.$$

These bounds thus speak to the concentration properties of eigenfunctions. When P is in some sense “far away” from containing geodesic segments, eigenfunctions have less tendency to concentrate near P. A work of Hassell-Tacy [9] proves bounds of this type in a semiclassical setting.

In the present work, we consider the development of the bounds (1.3) for $C^{1,\alpha}$ metrics with $0 \le \alpha \le 1$, allowing for Lipschitz regularity when $\alpha = 0$. As a corollary, we obtain bounds of this type (with a loss) for the Dirichlet and Neumann Laplacians on compact manifolds with boundary. Bounds of the form (1.3) when $n = 2$, $k = 1$ for manifolds with concave boundaries are due to Ariturk [1], provided Dirichlet conditions are imposed. However, the presence of gliding rays when the manifold possesses a point of convexity within the boundary complicates matters considerably.

Theorem 1.1. *Suppose* $\mathrm{g} \in C^{1,\alpha}$ *with* $0 \le \alpha \le 1$*, allowing for Lipschitz regularity when* $\alpha = 0$*. When* $k = n-1$ *and* $2 \le q \le \frac{2n}{n+1}$*, we have that for* $\delta = \frac{n-1}{4} - \frac{n-2}{2q}$

$$\|\Pi_\lambda f\|_{L^q(P)} \le C\lambda^{\delta(1+\sigma)}\|f\|_{L^2(M)}, \qquad \sigma = \frac{1-\alpha}{3+\alpha}. \tag{1.7}$$

Moreover, when $k = n-1$*,* $\frac{2n}{n+1} \le q \le \infty$ *or* $k \le n-2$ *we suppose that* $\delta = \frac{n-1}{2} - \frac{k}{q}$ *and* $\delta + \frac{\sigma}{q} < 1 + \alpha$ *with* σ *as above. In this case, the following bounds are satisfied,*

$$\|\Pi_\lambda f\|_{L^q(P)} \le C\lambda^{\delta+\frac{\sigma}{q}}\|f\|_{L^2(M)} \tag{1.8}$$

with C *replaced by* $C(\log\lambda)^{\frac{1}{2}}$ *when* $(k,q) = (n-2,2)$*. The admissibility condition on* δ*,* q *can be relaxed to* $\delta + \frac{\sigma}{q} \le 1 + \alpha$ *when* $\alpha = 0$ *or* $\alpha = 1$*.*

Furthermore, we will show improvements akin to (1.6) when $0 < \alpha \le 1$. For these metrics the Christoffel symbols are well defined and continuous on M by the usual coordinate formula

$$\Gamma^k_{ij} = \frac{1}{2}\mathrm{g}^{kl}\left(\partial_i \mathrm{g}_{jl} + \partial_j \mathrm{g}_{il} - \partial_l \mathrm{g}_{ij}\right)$$

(with the summation convention in effect). Hence there is also a well defined Levi-Civita connection associated to the metric g on M, mapping C^1 vector fields to continuous vector fields with the usual properties. In particular, given a smooth, embedded, codimension 1 submanifold of P, the scalar second fundamental form is well defined and if it is negative definite throughout P for a suitable choice of normal vector field, we shall call it "curved". We will see that in this case, the power of λ in (1.7) with $q = 2$ can be improved to $\frac{1}{6} + \frac{\sigma}{2}$ (which can be seen as strictly less than $\frac{1}{4}(1+\sigma)$ when $\sigma < \frac{1}{3}$).

Theorem 1.2. *Suppose* $\mathrm{g} \in C^{1,\alpha}$ *with* $0 < \alpha \leq 1$, *and that* P *is a "curved" codimension 1 submanifold as defined above. Then the following bounds are satisfied*

$$\|\Pi_\lambda f\|_{L^2(P)} \leq C\lambda^{\frac{1}{6}+\frac{\sigma}{2}} \|f\|_{L^2(M)}, \qquad \sigma = \frac{1-\alpha}{3+\alpha}. \tag{1.9}$$

Moreover, interpolating this bound with the $q = \frac{2n}{n-1}$ *case of* (1.7) *yields an improvement of that estimate for* $2 \leq q < \frac{2n}{n-1}$.

Following [19], we will show that for each theorem, the $0 \leq \alpha < 1$ case follows from the $\alpha = 1$ case by rescaling methods. This involves dilating coordinates so that sets of diameter $\approx \lambda^{-\sigma}$ in P have diameter ≈ 1 in the new coordinates. Since the metric can be approximated by one with $C^{1,1}$ regularity here, the bounds from the $\alpha = 1$ case can then be applied. In the original coordinates, this then implies that the estimates (1.7), (1.8), (1.9) hold with $\sigma = 0$ over sets of diameter $\approx \lambda^{-\sigma}$. By incorporating the flux estimates in [19], it can then be seen that Theorems 1.1 and 1.2 follow by taking a sum over all such sets.

The bounds (1.1) for $C^{1,1}$ metrics in [18] (and those for manifolds with boundary in [17]) follow by wave equation methods. Specifically, square function estimates are developed for solutions to the wave equation on these manifolds, bounding the $L^q(M)$ norm of the square function

$$x \mapsto \left(\int_0^1 |u(t,x)|^2 \, dt \right)^{\frac{1}{2}}, \qquad \text{where } (\partial_t^2 - \Delta_{\mathrm{g}})u = 0.$$

As will be seen below, spectral clusters above naturally give rise to solutions to the wave equation, these estimates imply bounds on the $\Pi_\lambda f$. Square function estimates were first proved in [13] for smooth metrics, using that Fourier integral operators can be used to invert the equation. However, when $\mathrm{g} \in C^{1,1}$, the roughness of the metric means that these methods are inapplicable, so a crucial development in [18] is the construction of a suitable parametrix using wave packet methods. The resulting approximate solution operators can be thought of as generalized Fourier integral operators where the associated canonical relation satisfies the curvature condition in [13].

We follow the same strategy here, essentially proving bounds on the $L^q(P)$ norm of the square function above. Once again, the roughness of the metric means that we are led to use wave packet methods to construct a parametrix. In this case, the canonical relations which arise naturally have folding singularities. In Theorem 1.1, the relation has a one-sided fold and in Theorem 1.2 the relation essentially has a two-sided fold. There is a significant body of work on $L^2 \to L^q$ bounds for Fourier integral operators with folding singularities, see [8], [10], [12], [14], [5] (the first one treating one sided folds). A key technical development in the present work is that the operators arising from the wave packet transform satisfy the desired square

function estimates in spite of the inapplicability of these results for Fourier integral operators. Nonetheless, the approach taken here is in part inspired by these works.

Notation. We use C^α to Hölder class of order α. Moreover $C^{1,\alpha}$ will denote the class of metrics or functions whose first derivative is in C^α, taking the artificial convention that Lipschitz regularity is allowable when $\alpha = 0$. In what follows, $X \lesssim Y$ will denote that $X \leq CY$ for some implicit constant C which is in some sense uniform, though when used in decay estimates, it may depend on the order N. Similarly, $X \approx Y$ will denote that $X \lesssim Y$ and $Y \lesssim X$. We use d as the differential which carries scalar functions to covector fields and vectors into matrices in the natural way. Lastly, given a vector $x \in \mathbb{R}^n$, x' and x'' will typically denote a vector in $\mathbb{R}^l$, $l < n$, formed by taking a subcollection of the components of x. The nature of this subcollection may vary depending on the section.

Remark on admissibility conditions. The admissibility condition $\delta + \frac{\sigma}{q} < 1+\alpha$ (with equality allowed when $\alpha = 0, 1$) arises in §2 where elliptic regularity is used to show that when a cluster $\Pi_\lambda f$ is considered in a coordinate system, the high frequency components (with respect to the Fourier transform) satisfy better bounds than those near frequency λ. However, it can be checked that the condition $\delta < \frac{1}{2}$ is always satisfied when $k = n-1$ and $2 \leq q \leq \frac{2n}{n-1}$ and that $\delta < \frac{5}{6}$ holds for sufficiently small $q > 2$ when $k = 2$ ensuring that in many relevant cases, the admissibility condition is satisfied. On the other hand, Smith [19, p. 969] showed that the bound $\|\Pi_\lambda f\|_{L^\infty(M)} \lesssim \lambda^{\frac{n-1}{2}} \|f\|_{L^2(M)}$ holds whenever g is Lipschitz. The key observation here is that one can write $\Pi_\lambda f = \exp(-\lambda^{-2}\Delta_{\mathrm{g}})\Pi_\lambda \tilde{f}$ with $\|\Pi_\lambda \tilde{f}\|_{L^2(M)} \approx \|\Pi_\lambda f\|_{L^2(M)}$. The $L^\infty(M)$ bounds then follow by combining Saloff-Coste's [16] Gaussian upper bounds on the heat kernel with Smith's $L^{\frac{2(n+1)}{n-1}}(M)$ bounds on $\Pi_\lambda f$. However, the same argument gives the continuity of each $\Pi_\lambda f \in L^2(M)$ since the fixed time heat kernel is continuous on $M \times M$ (as observed in [16, §6]). Thus Smith's L^∞ bounds on spectral clusters imply L^∞ bounds on their restrictions and this can be interpolated with the $L^q(P)$ bounds for submanifolds of low codimension to see that in many cases, the admissibility conditions can be relaxed. This also ensures that the restrictions are well-defined.

Remark on the optimality of (1.7). As noted above in (1.2), the examples in [20], [21] show that the bounds of Smith [19] establishing $L^q(M)$ bounds are sharp for small values of $q > 2$. We comment here that the same examples show that the bounds (1.7) in Theorem 1.1 are sharp as well. Indeed, the examples in [20] produce metrics of $C^{1,\alpha}$ regularity and associated spectral clusters f_λ which are concentrated in a tube of length 1 and diameter $\lambda^{-\frac{2}{3+\alpha}}$, that is, a set of the form

$$|x_1| \lesssim 1 \qquad |(x_2, \dots, x_n)| \lesssim \lambda^{-\frac{2}{3+\alpha}}. \tag{1.10}$$

Therefore if we take P to be defined by $x_n = 0$, we see that the rapid decay outside of this set implies that

$$\frac{\|f_\lambda\|_{L^q(P)}}{\|f_\lambda\|_{L^2(M)}} \approx \lambda^{\frac{2}{3+\alpha}(\frac{n-1}{2} - \frac{n-2}{q})}.$$

However, $\frac{1}{2}(\sigma+1) = \frac{2}{3+\alpha}$, showing that the exponent simplifies to $\delta(1+\sigma)$ and hence the bound (1.7) is optimal.

2. Microlocal reductions

In this section, we will reduce the main theorems to proving square function estimates for frequency localized solutions to a hyperbolic pseudodifferential equation. We follow an approach due to H. Smith [19] (see also [3]). The needed reductions are fairly common to both theorems (with some exceptions when P is curved), so we begin by treating all cases at the same time. It is thus convenient to take the convention that $\delta(\sigma)$ is defined by taking the power of λ appearing in (1.7), (1.8), or (1.9), realizing that in all cases $\delta(0)$ denotes the power without loss of derivatives. Moreover, the admissibility conditions mean that if $\sigma > 0$ and $\delta(\sigma) > 1$, $\delta(\sigma) - 1 < \alpha$ (respectively $\delta(\sigma) - 1 \leq \alpha$ when $\alpha = 0, 1$).

Throughout these preliminary reductions, we will make use of the fact that when $k < n$, we have the following embedding for traces in $\mathbb{R}^k \times \{0\}$, $\{0\} \in \mathbb{R}^{n-k}$

$$H^{\frac{n}{2}-\frac{k}{q}}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^k \times \{0\}), \tag{2.1}$$

which can be seen by first applying Sobolev embedding on $\mathbb{R}^k \times \{0\}$, then using the trace theorem for L^2 based Sobolev spaces. The estimates in Theorems 1.1 and 1.2 thus exhibit a gain relative to Sobolev embedding. The gain is largest when $q = 2$, in which case there is a gain of a 1/4 or 1/3 of a derivative when $k = n - 1$ depending on whether the submanifold is curved and a gain of 1/2 a derivative (up to a possible logarthmic correction) when $k \leq n - 2$.

It suffices to prove the main theorem for a spectral cluster f satisfying $f = \Pi_\lambda f$. We begin by observing that f satisfies the following bounds in Sobolev spaces defined by the spectral resolution of Δ_{g}

$$\|(\Delta_{\mathrm{g}} + \lambda^2) f\|_{H^s(M)} + \|df\|_{H^s(M)} \lesssim \lambda^{s+1} \|f\|_{L^2(M)}.$$

It thus suffices to prove bounds on f of the following form

$$\|f\|_{L^q(P)} \lesssim \sum_i \lambda^{\delta(\sigma)-1-s_i} \left(\lambda \|f\|_{H^{s_i}(M)} + \|df\|_{H^{s_i}(M)} + \|(\Delta_{\mathrm{g}} + \lambda^2) f\|_{H^{s_i}(M)}\right) \tag{2.2}$$

where a sum is taken over a finite collection of $0 \leq s_i \leq 1$.

Multiplication by any smooth bump function ψ preserves $H^1(M)$, and by interpolation $H^s(M)$ for any $s \in [0, 1]$. Therefore by taking a partition of unity on M, it suffices to prove (2.2) with f replaced by ψf, where ψ is supported in a suitable coordinate chart which intersects P. Specifically, we will take slice coordinates so that P is identified with $\mathbb{R}^k \times \{0\}$. Furthermore, by taking a sufficiently fine partition of unity and dilating coordinates, we may assume that for some c_0 sufficiently small,

$$\|\mathrm{g}^{ij} - \delta_{ij}\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq c_0. \tag{2.3}$$

By elliptic regularity (see e.g. Gilbarg-Trudinger [6, Theorem 8.10, Theorem, 9.11]) and interpolation we have that for any g supported in this coordinate chart $\|g\|_{H^s(M)} \approx \|g\|_{H^s(\mathbb{R}^n)}$ for $s \in [0, 2]$. Next we observe that in coordinates within $\mathrm{supp}(\psi)$, f satisfies an equation of the form

$$\mathrm{g}d^2 f + \lambda^2 f = w, \qquad \mathrm{g}d^2 f = \sum_{1 \leq i,j \leq n} \mathrm{g}^{ij} \partial^2_{ij} f \tag{2.4}$$

where w is a sum consisting of $(\Delta_{\mathrm{g}} + \lambda^2) f$ and products of the form $a \cdot \partial_j f$, with $a \in C^\alpha$ (or L^∞, $C^{0,1}$ when $\alpha = 0, 1$ respectively) in turn a product of functions

of the form g^{ij}, $\sqrt{\det \mathrm{g}_{ij}}$ or their first derivatives. Hence multiplication by these functions preserves $H^s(\mathbb{R}^n)$ for $s = 0$ and $s \in [0, \alpha)$ when $\alpha > 0$ (respectively $s \in [0, 1]$ when $\alpha = 1$) meaning that that for any such s

$$\|w\|_{H^s(\mathbb{R}^n)} \lesssim \|f\|_{H^s(M)} + \|df\|_{H^s(M)} + \|(\Delta_{\mathrm{g}} + \lambda^2)f\|_{H^s(M)}.$$

Furthermore, elliptic regularity (see e.g. [6, Theorem, 9.11]) also gives that

$$\|d^2 f\|_{L^2(\mathbb{R}^n)} \lesssim \|\Delta_{\mathrm{g}} f\|_{L^2(\mathbb{R}^n)} + \|df\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

Moreover, when $\delta(\sigma) > 1$ (which only occurs when $\alpha > 0$), we have that

$$\|[\mathrm{g}^{ij}, \langle D\rangle^{\delta(\sigma)-1}]\partial^2_{ij} f\|_{L^2(\mathbb{R}^n)} + \|[\partial_i \mathrm{g}, \langle D\rangle^{\delta(\sigma)-1}]\partial_j f\|_{L^2(\mathbb{R}^n)} \lesssim \|df\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)}, \tag{2.6}$$

where $\langle D\rangle$ denotes the Fourier multiplier with symbol $(1 + |\xi|^2)^{\frac{1}{2}}$. This means that we may replace L^2 by $H^{\delta(\sigma)-1}$ in (2.5). Indeed, the bound on the first term in (2.6) follows as a consequence of the Coifman-Meyer commutator theorem (see e.g. [26, Proposition 3.6B]) and the second follows since the admissibility condition on $\delta(\sigma)$ implies that multiplication by $\partial_i \mathrm{g}$ preserves $H^{\delta(\sigma)-1}(\mathbb{R}^n)$.

With this in mind, we define the following norm when $\delta(\sigma) \le 1$

$$|||f||| := \|f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1}\|df\|_{L^2(\mathbb{R}^n)} + \lambda^{-2}\|d^2 f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1}\|w\|_{L^2(\mathbb{R}^n)}.$$

When $\delta(\sigma) > 1$, we define

$$\begin{aligned} |||f||| := &\sum_{j=0}^{2} \lambda^{-j}\|d^j f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1}\|w\|_{L^2(\mathbb{R}^n)} \\ &\qquad + \lambda^{-(\delta(\sigma)-1)} \left(\sum_{j=0}^{2} \lambda^{-j}\|d^j f\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)} + \lambda^{-1}\|w\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)} \right). \end{aligned}$$

Given the observations above, it now suffices to show that

$$\|f\|_{L^q(\mathbb{R}^k \times \{0\})} \lesssim \lambda^{\delta(\sigma)} |||f|||. \tag{2.7}$$

Without loss of generality, we may assume that f is supported in a cube of sidelength 1 centered at the origin and that the metric is defined over a cube of sidelength 8 centered at the origin. Hence we may smoothly extend the metric g so that it is defined over all of $\mathbb{R}^n$ and equal to the flat metric for $|x|$ sufficiently large without altering the equation for f. Given $r > 0$, we let $S_r = S_r(D)$ denote a Fourier multiplier which applies a smooth cutoff to frequencies $|\xi| \le r$ and define $\mathrm{g}_\lambda = S_{c^2\lambda}\mathrm{g}$ where $c > 0$ will be taken to be sufficiently small. Since

$$\|\mathrm{g}_\lambda - \mathrm{g}\|_{L^\infty} \lesssim \lambda^{-1} \tag{2.8}$$

we may replace g by g_λ in (2.4) when $\delta(\sigma) \le 1$ as the error can be absorbed into the right hand side of (2.7). The same holds when $1 < \delta(\sigma)$ is admissible, which can be seen by using the similar bound $\|\mathrm{g}_\lambda - \mathrm{g}\|_{C^\alpha} \lesssim \lambda^{-1}$ and the fact that multiplication by a C^α function preserves $H^{\delta(\sigma)-1}(\mathbb{R}^n)$.

We now write f as $f = f_{<\lambda} + f_\lambda + f_{>\lambda}$ where $f_{<\lambda} = S_{c\lambda} f$ and $f_{>\lambda} = f - S_{c^{-1}\lambda} f$. Observe that when $s = 0$

$$\|[S_{c\lambda}, \mathrm{g}_\lambda]\|_{H^s \to H^s} + \|[S_{c^{-1}\lambda}, \mathrm{g}_\lambda]\|_{H^s \to H^s} \lesssim \lambda^{-1} \tag{2.9}$$

which follows from simple bounds on the kernel of the commutators. When $1 < \delta(\sigma)$ is admissible, the same holds with $s = \delta(\sigma) - 1$. Indeed, we have that $\lambda S_{c\lambda}$ (and similarly $\lambda S_{c^{-1}\lambda}$) defines an operator in $S^1_{1,0}$ hence the symbolic calculus

gives $[\lambda S_{c\lambda}, \mathrm{g}_\lambda] \in C^\alpha S^0_{1,0}$ (in the notation of [26]). The claim then follows by [26, Proposition 2.1D] or by commuting with derivatives when $\alpha = 1$. Defining $w_{<\lambda} := \mathrm{g}_\lambda d^2 f_{<\lambda} + \lambda^2 f_{<\lambda}$, $w_{>\lambda} := \mathrm{g}_\lambda d^2 f_{>\lambda} + \lambda^2 f_{>\lambda}$ we have

$$\|w_{<\lambda}\|_{H^s(\mathbb{R}^n)} + \|w_{>\lambda}\|_{H^s(\mathbb{R}^n)} \lesssim \lambda^{-1}\|d^2 f\|_{H^s(\mathbb{R}^n)} + \|w\|_{H^s(\mathbb{R}^n)} \tag{2.10}$$

for $s = 0$ and for $s = \delta(\sigma) - 1$ when the latter quantity is positive.

To bound $f_{<\lambda}$, $f_{>\lambda}$ we use arguments from [19, Corollary 5]. Since $\|\mathrm{g}_\lambda d^2 f_{<\lambda}\|_{L^2} \lesssim (c\lambda)^2 \|f_{<\lambda}\|_{L^2}$, (2.1) and the equation give the stronger estimate

$$\|f_{<\lambda}\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{\frac{n}{2}-\frac{k}{q}}\|f_{<\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{\frac{n}{2}-\frac{k}{q}-2}\|w_{<\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \lambda^{\frac{n}{2}-\frac{k}{q}-1}|||f|||.$$

For the high frequency term $f_{>\lambda}$, we use that when $s \geq 0$,

$$\lambda^2\|f_{>\lambda}\|_{H^s(\mathbb{R}^n)} + \lambda\|df_{>\lambda}\|_{H^s(\mathbb{R}^n)} \lesssim c\|d^2 f_{>\lambda}\|_{H^s(\mathbb{R}^n)}.$$

This bound with $s = 0$ can be combined with elliptic regularity to obtain

$$\|d^2 f_{>\lambda}\|_{L^2(\mathbb{R}^n)} \lesssim \|w_{>\lambda}\|_{L^2(\mathbb{R}^n)}. \tag{2.11}$$

When $\frac{n}{2} - \frac{k}{q} \leq 2$, (2.1) yields a gain of at least 1/2 of a derivative in the estimate for $f_{>\lambda}$. The case $\frac{n}{2} - \frac{k}{q} > 2$ only arises when $\alpha > 0$ and $\delta(\sigma) = \frac{n-1}{2} - \frac{k}{q} + \frac{\sigma}{q}$, and in this case we use (2.6) (with g_λ replacing g) to bootstrap the elliptic regularity estimate, which yields a similar gain for $f_{>\lambda}$ since

$$\|f_{>\lambda}\|_{H^{\delta(\sigma)+1}(\mathbb{R}^n)} \lesssim \|w_{>\lambda}\|_{H^{\delta(\sigma)-1}(\mathbb{R}^n)} \lesssim |||f|||.$$

We are now reduced to proving bounds on f_λ. Reasoning as in (2.9), we have that $|||f_\lambda||| \lesssim |||f|||$. We now impose a further microlocal decomposition of the function, writing $f_\lambda = f_{\lambda,T} + f_{\lambda,N}$, where $\widehat{f}_{\lambda,T}$ is localized to directions tangent to the submanifold and $\widehat{f}_{\lambda,N}$ is localized to normal directions. Specifically, we write $f_{\lambda,N} = \sum_{j=k+1}^n f_{\lambda,j}$ where $f_{\lambda,j}$ is frequency localized to a set of the form

$$\operatorname{supp}(\widehat{f_{\lambda,j}}) \subset \{\xi : \lambda \approx |\xi|,\, |\xi_j| \gtrsim \varepsilon^{-1}|(\xi_1, \dots, \xi_j, \xi_{j+1}, \dots, \xi_n)|\},$$

with ε suitably small. Using (2.9) again, we have that

$$\|\mathrm{g}_\lambda d^2 f_{\lambda,j} + \lambda^2 f_{\lambda,j}\|_{L^2(\mathbb{R}^n)} \lesssim |||f_\lambda|||. \tag{2.12}$$

With this in mind, the flux estimates of Smith [19, p.974], give

$$\|f_{\lambda,j}\|_{L^\infty_{x_j} L^2_{x'}} \lesssim |||f_\lambda||| \tag{2.13}$$

where x' denotes the vector consisting of every component in $\mathbb{R}^n$ but x_j. Combining this with the $n-1$ dimensional version of (2.1) on the hyperplane $x_j = 0$ if necessary, we have

$$\|f_{\lambda,j}\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{\frac{n-1}{2}-\frac{k}{q}}\|f_{\lambda,j}\|_{L^2(x_j=0)} \lesssim \lambda^{\delta(\sigma)}|||f_\lambda|||.$$

We now further decompose $f_{\lambda,T}$ as $f_{\lambda,T} = \sum_j f_{\lambda,\omega_j}$ where $\{\omega_j\}$ is a finite collection of unit vectors and $\operatorname{supp}(\widehat{f_{\lambda,\omega_j}})$ lies in a small conic set containing ω_j. Without loss of generality, it suffices to treat the case $\omega_j = -e_1 = (-1, 0, \dots, 0)$. Recalling (2.12) and simplifying notation it now suffices to prove $\|f_\lambda\|_{L^q(\mathbb{R}^k\times\{0\})} \lesssim \lambda^{\delta(\sigma)}|||f_\lambda|||$ for f_λ satisfying

$$\operatorname{supp}(\widehat{f_\lambda}) \subset \{\xi : |\xi/|\xi| - (-e_1)| \lesssim \varepsilon\}. \tag{2.14}$$

As a consequence of (2.13) with $x_j = x_1$ and Hölder's inequality, we have that if S_R is a slab of the form $S_R = \{x : |x_1 - r| \le R\}$ for some r

$$\|f_\lambda\|_{L^2(S_R)} \lesssim R^{\frac12} |||f_\lambda|||. \tag{2.15}$$

Set $\rho = \frac{n-1}{2} - \frac{k}{q}$ so that $\rho - \delta(0) = \delta(0) - \frac1q$ when $k = n-1$, $2 \le q \le \frac{2n}{n-1}$ and $\rho = \delta(0)$ in all other cases of Theorem 1.1. Given a cube Q_R of sidelength $R = \lambda^{-\sigma}$ which intersects $\mathbb{R}^k \times \{0\}$, we let Q_R^* denote its double, and also set $w_\lambda := \mathrm{g}_\lambda d^2 f_\lambda + \lambda^2 f_\lambda$. We claim that Theorem 1.1 now follows from the bound

$$\|f_\lambda\|_{L^q((\mathbb{R}^k\times\{0\})\cap Q_R)} \lesssim \lambda^{(1-\sigma)\delta(0)} R^{-\rho} \left(R^{-\frac12}\|f_\lambda\|_{L^2(Q_R^*)} + R^{\frac12}\lambda^{-1}\|w_\lambda\|_{L^2(Q_R^*)}\right). \tag{2.16}$$

Moreover, Theorem 1.2 will follow from taking $q = 2$, $\delta(0) = 1/6$ here when P is curved (as $\rho = 0$ in this case). Indeed, if these bounds hold, we may sum over the cubes Q_R contained in S_R which intersect $\mathbb{R}^k \times \{0\}$ to obtain

$$\|f_\lambda\|_{L^q((\mathbb{R}^k\times\{0\})\cap S_R)} \lesssim \lambda^{(1-\sigma)\delta(0)+\sigma\rho} \left(R^{-\frac12}\|f_\lambda\|_{L^2(S_R^*)} + R^{\frac12}\lambda^{-1}\|w_\lambda\|_{L^2(S_R^*)}\right).$$

Recalling (2.15), the right hand side is bounded by $\lambda^{(1-\sigma)\delta(0)+\sigma\rho}|||f_\lambda|||$. Given the previous observations on ρ, the desired bound on f_λ then follows by taking a sum over the $\mathcal{O}(R^{-1})$ slabs S_R in $|x_1| \le 3/4$ and the rapid decay property

$$|f_\lambda(x)| \lesssim (\lambda|x|)^{-N}\|f_\lambda\|_{L^2(\mathbb{R}^n)} \qquad \text{for } \max_j |x_j| \ge 3/4. \tag{2.17}$$

The latter is a consequence of our assumption that f is supported in a cube of sidelength 1 at the origin, which implies that f_λ is concentrated in a λ^{-1} neighborhood of this cube.

We now dilate variables $x \mapsto Rx$, set $\mu := R\lambda$, and make the slight abuse of notation that $f_\mu(x) = f_\lambda(Rx)$. We will see that this reduces the general bounds to those without a loss of derivatives, and hence we will take $\delta = \delta(0)$ below. Indeed, rescaling the bound (2.16) gives

$$\|f_\mu\|_{L^q((\mathbb{R}^k\times\{0\})\cap Q)} \lesssim \mu^\delta \left(\|f_\mu\|_{L^2(Q^*)} + \mu^{-1}\|\mathrm{g}_\mu d^2 f_\mu + \mu^2 f_\mu\|_{L^2(Q^*)}\right) \tag{2.18}$$

(when P is curved, rescaling yields the same with $q = 2$ and $\delta = (1+\beta)/6$ where $\beta = \sigma/(1-\sigma)$). Here Q is now a cube of sidelength 1, which we may take to be centered at the origin, and $\mathrm{g}_\mu(x) := \mathrm{g}_\lambda(Rx)$. We now have that if $\mathrm{g}_{\mu^{1/2}} := S_{c^2\mu^{1/2}}\mathrm{g}_\mu$, then (cf. (2.3))

$$\|\mathrm{g}_\mu - \mathrm{g}_{\mu^{1/2}}\|_{L^\infty} \lesssim c_0\mu^{-1} \tag{2.19}$$

and we may replace g_μ by $\mathrm{g}_{\mu^{1/2}}$ in (2.18) since the error can be absorbed in to the right hand side. The metric $\mathrm{g}_{\mu^{1/2}}$ has C^2 regularity, namely

$$\|\mathrm{g}^{ij}_{\mu^{1/2}} - \delta_{ij}\|_{C^2} \lesssim c_0. \tag{2.20}$$

Furthermore,

$$\|\partial^\alpha \mathrm{g}^{ij}_{\mu^{1/2}}\|_{C^2} \le \mu^{\frac12(|\alpha|-2)}, \qquad |\alpha| \ge 2. \tag{2.21}$$

We will prove the bound (2.18) by wave equation methods. Let $u_\mu(t,x) = \cos(t\mu) f_\mu(x)$. It suffices to show that if $F_\mu = (\partial_t^2 - \mathrm{g}_{\mu^{1/2}} d^2)u_\mu$

$$\|u_\mu\|_{L^q((\mathbb{R}^k\times\{0\})\cap Q; L^2(-\frac12,\frac12))} \lesssim \mu^\delta \left(\|u_\mu(0,\cdot)\|_{L^2(Q^*)} + \mu^{-1}\|F_\mu\|_{L^2((-1,1)\times Q^*)}\right).$$

Now let $\psi(t,x)$ denote a smooth cutoff identically one on $(-\frac{1}{2},\frac{1}{2})^{n+1}$ and supported in $(-\frac{3}{4},\frac{3}{4})^{n+1}$. Replacing u_μ by ψu_μ, and similarly for F_μ, it suffices to show that

$$\|u_\mu\|_{L^q((-\frac{1}{2},\frac{1}{2})\times\mathbb{R}^{k-1}\times\{0\};L^2(\mathbb{R}))} \lesssim \mu^\delta\left(\|u_\mu(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \mu^{-1}\|F_\mu\|_{L^2(\mathbb{R}^{n+1})}\right) \tag{2.22}$$

since energy estimates bound the error terms which arise when commuting $(\partial_t^2 - \mathrm{g}_{\mu^{1/2}}d^2)$ with ψ. Next we let $\Gamma^\pm_\mu(\tau,\xi)$ be smooth cutoffs to the regions

$$\{(\tau,\xi) : \pm\tau \approx \mu, -\xi_1 \approx \mu, |\xi| \approx \mu\} \tag{2.23}$$

and supported in a slightly larger set. Let $u^\pm_\mu = \Gamma^\pm_\mu(D_{t,x})u_\mu$. By [18, Lemma 2.3] and the localization of f_μ, we see that elliptic regularity and (2.1) yields an estimate on $u_\mu - u^+_\mu - u^-_\mu$ with a gain of at least a half a derivative relative to the right hand side of (2.22).

Given (2.20), we have that for (τ,ξ) in the regions (2.23), $\mathrm{g}^{ij}_{\mu^{1/2}}\xi_i\xi_j - \tau^2$ defines a quadratic in ξ_1 with two real roots and hence we may write

$$\mathrm{g}^{ij}_{\mu^{1/2}}(x)\xi_i\xi_j - \tau^2 = \mathrm{g}^{11}_{\mu^{1/2}}(x)\left(\xi_1 + q^-(x,\tau,\xi')\right)\left(\xi_1 - q^+(x,\tau,\xi')\right) \tag{2.24}$$

with $q^\pm > 0$ and homogeneous of degree 1 for such (τ,ξ). We further regularize these symbols taking $p^\pm(\cdot,\tau,\xi') = S_{c^2\mu^{\frac{1}{2}}}q^\pm(\cdot,\tau,\xi')$. By the elliptic regularity argument in [18, Lemma 2.4], the function u_μ satisfies

$$\left(-i\partial_{x_1} + p^-(x,D_{t,x'})\right)u^\pm_\mu = G^\pm_\mu, \tag{2.25}$$

with $\|G^\pm_\mu\|_{L^2(\mathbb{R}^{n+1})}$ bounded by the terms in parentheses on right hand side of (2.22). Moreover, akin to (2.17), we have the rapid decay property

$$|u^\pm_\mu(t,x)| \lesssim (\mu|(t,x)|)^{-N}\|u_\mu\|_{L^2(\mathbb{R}^{n+1})}, \quad \text{for } \max(|t|,|x_1|,\dots,|x_n|) \geq 1. \tag{2.26}$$

Thus by energy estimates it can be seen that

$$\|u^\pm_\mu\|_{L^2(\mathbb{R}^{n+1})} \lesssim \|u_\mu(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \mu^{-1}\|\partial_t u_\mu(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \mu^{-1}\|G_\mu\|_{L^2(\mathbb{R}^{n+1})}$$

since the right hand side is compactly supported. It now suffices to show that

$$\|u^\pm_\mu\|_{L^q((-\frac{1}{2},\frac{1}{2})\times\mathbb{R}^{k-1}\times\{0\};L^2(\mathbb{R}))} \lesssim \mu^\delta\left(\|u^\pm_\mu\|_{L^2(\mathbb{R}^{n+1})} + \mu^{-1}\|G^\pm_\mu\|_{L^2(\mathbb{R}^{n+1})}\right). \tag{2.27}$$

2.1. **Special reductions for curved submanifolds.** The reductions in the case where P is curved follows essentially the same as the general case, though there are some additional considerations that we discuss here. We return to the metric $\mathrm{g}_\lambda(x)$, which we considered before the dilation of variables $x \mapsto Rx$ had occurred. In this case, if N denotes a suitable vector field of unit length such that $N_p \perp P$ for any $p \in P$ which lies in the cube of sidelength 4 centered at the origin in our coordinate system, we can assume that there exists a uniform constant c_1 such that for $n-1$ vectors $(X^1,\dots,X^{n-1})$

$$-\sum_{1\leq i,j\leq n-1}\langle N,\nabla_{\partial_i}\partial_j\rangle_{\mathrm{g}_\lambda}X^iX^j \geq 8c_1\left((X^1)^2+\cdots+(X^{n-1})^2\right). \tag{2.28}$$

where $\langle\cdot,\cdot\rangle_{\mathrm{g}_\lambda}$ denotes the inner product determined by g_λ and ∇ denotes Levi-Civita connection. Indeed, the assumption on P ensures that such an estimate holds for a suitable choice of N when g_λ is replaced by g as $\partial_1,\dots,\partial_{n-1}$ span the tangent space to P in our coordinate system. This implies the bound above for sufficiently large λ by the inequalities (2.8) and $\|\mathrm{g}-\mathrm{g}_\lambda\|_{C^1} \lesssim \lambda^{-\alpha}$. We now observe that as before, that it suffices to show (2.16) with $\delta(0) = \frac{1}{6}$, $q = 2$ for cubes Q_R contained in the cube of sidelength 4 centered at the origin (cf. (2.17)). Now take the same

dilation of variables $x \mapsto Rx$ with $R = \lambda^{-\sigma}$. Since $\nabla_{\partial_i}\partial_j$ is determined by the Christoffel symbols, we have an inequality of the form (2.28) with g_λ replaced by $\mathrm{g}_{\mu^{1/2}}$ and $8c_1$ replaced by $4\mu^{-\beta}c_1$, where $\beta = \frac{\sigma}{1-\sigma}$ (cf. (2.19)). It now suffices to show (2.18) with $\delta = \frac{1}{6}(1+\beta)$, $q = 2$ for a cube Q of sidelength 1 centered at the origin.

We now make a further change of coordinates, taking Fermi coordinates $x_1, \dots, x_n$ near P (see [7, §2.1] for the construction and properties of such coordinates). In this coordinate system, x_n not only is a defining function but determines the geodesic distance from the point x to P. Since the metric has C^2 regularity and the coordinates are defined using the exponential map, we know that the coordinates exist in a neighborhood of the double Q^* provided c_0 is taken sufficiently small in (2.20). Furthermore, ∂_n defines a vector field of unit length and ∂_n is is orthogonal to surfaces of the form $x_n = c$, where c is a constant (see [7, Corollary 2.13]). Hence in this coordinate system the metric has a block diagonal form in that $\mathrm{g}_{\mu^{1/2}}(\partial_i, \partial_n) = \delta_{in}$.

Proposition 2.1. *In the given Fermi coordinates, $-\partial_n \mathrm{g}^{ij}_{\mu^{1/2}}$ defines a positive definite quadratic form on $n-1$ vectors $(\xi_1, \dots, \xi_{n-1})$ in that*

$$-\sum_{1\le i,j\le n-1} \partial_n \mathrm{g}^{ij}_{\mu^{1/2}}(x)\xi_i\xi_j \ge 2c_1\mu^{-\beta}|\xi'|^2, \qquad \beta = \frac{\sigma}{1-\sigma}. \tag{2.29}$$

Proof. For simplicity, we suppress the $\mu^{1/2}$ in the subscripts below. Begin by observing that since $[\partial_i, \partial_n] = 0$, the torsion-free property of the connection ensures that for any $i = 1, \dots, n-1$, $\nabla_{\partial_n}\partial_i = \nabla_{\partial_i}\partial_n$. Combining this with the compatibility of the connection, we have that at points p on the submanifold

$$\partial_n \mathrm{g}_{ij} = \langle \nabla_{\partial_i}\partial_n, \partial_j\rangle_{\mathrm{g}} + \langle \partial_i, \nabla_{\partial_j}\partial_n\rangle_{\mathrm{g}} = -2\langle \partial_n, \nabla_{\partial_i}\partial_j\rangle_{\mathrm{g}}$$

with the last property a consequence of the Weingarten equation and the fact that ∂_n is the unit normal to the submanifold. Recall that the last quantity defines a positive definite quadratic form with eigenvalues bounded below by $4\mu^{-\beta}c_1$.

We now write any nonzero vector $(\xi_1, \dots, \xi_{n-1})$ as $\xi_i = \mathrm{g}_{il}X^l$ with summation convention in effect. Making use of the identity $-(\partial_n \mathrm{g}^{ij})\mathrm{g}_{il} = (\partial_n \mathrm{g}_{il})\mathrm{g}^{ij}$, the inequality (2.29) follows from the observations above and the identities

$$-\partial_n \mathrm{g}^{ij}\xi_i\xi_j = -(\partial_n \mathrm{g}^{ij})\mathrm{g}_{il}X^l \mathrm{g}_{jm}X^m = (\partial_n \mathrm{g}_{il})\mathrm{g}^{ij}\mathrm{g}_{jm}X^l X^m = \partial_n \mathrm{g}_{ml}X^l X^m.$$

□

The change of coordinates means $\widehat{f_\mu}$ is no longer localized to frequencies near μ. However, we may reason as before to see that the high and low frequency components away from μ satisfy better estimates than those localized to frequencies near μ. Hence, we may continue to assume that $\widehat{f_\mu}$ is supported near μ. Moreover, that $\mathrm{g}_{\mu^{1/2}}$ is no longer localized to frequencies less than $c^2\mu^{\frac{1}{2}}$. However, truncating the metric again to these frequencies does not alter the results in Proposition 2.1 (though $2c_1$ should be replaced by c_1) nor the properties established above. Defining $u^\pm_\mu$, $G^\pm_\mu$, $p^\pm$ as above and recalling the rapid decay property (2.26), the same considerations as before mean it suffices to show that over $\widetilde{Q} := (-1,1)^{n+1}$,

$$\|u_\mu\|_{L^2(\widetilde{Q})} \lesssim \mu^{\frac{1}{6}(1+\beta)} \left(\|u_\mu\|_{L^2(\mathbb{R}^{n+1})} + \mu^{-1}\|G_\mu\|_{L^2(\mathbb{R}^{n+1})}\right). \tag{2.30}$$

Theorem 2.2. *Suppose that for $|r| \le 1$, $(t(r), x(r), \tau(r), \xi(r))$ is a null bicharacteristic of $\xi_1 + p^-(x, \tau, \xi')$, lying in (2.23) (with the minus sign taken). By this we mean that the curve is contained in the zero set of the function and solves*

$$\begin{aligned} \frac{dt}{dr} &= \partial_\tau p^-(x,\tau,\xi') & \frac{d\tau}{dr} &= 0 \\ \frac{dx_1}{dr} &= 1 & \frac{d\xi_1}{dr} &= -\partial_{x_1} p^-(x,\tau,\xi') \\ \frac{dx'}{dr} &= d_{\xi'} p^-(x,\tau,\xi') & \frac{d\xi'}{dr} &= -d_{x'} p^-(x,\tau,\xi'). \end{aligned} \tag{2.31}$$

Assume also that $\tau(0)^2 + |\xi'(0)|^2 = 1$ and the curve lies in a set where (2.29) holds. Then the $\xi_n(r)$ component satisfies

$$\frac{1}{2} c_1 \mu^{-\beta} |r| \le \xi_n(r) - \xi_n(0) \le 2 c_0 \mu^{-\beta} |r|. \tag{2.32}$$

Furthermore, the component $x_n(r)$ satisfies the bounds

$$\left| x_n(r) - x_n(0) - \frac{\xi_n(0) r}{\mathrm{g}^{11}(x(0))(\xi_1(0) - p^+(0))} \right| \lesssim c_0 \left(\left(\mu^{-\beta} + |\xi_n(0)|\right) r^2 + \mu^{-\frac{1}{2}} |r| \right) \tag{2.33}$$

where we make the slight abuse of notation $p^+(0) = p^+(x(0), \tau(0), \xi(0))$.

Proof. First observe the following bounds, with uniform implicit constants provided $\tau^2 + |\xi'|^2 = 1$ and $|\gamma| \le 1$ (cf. (2.19))

$$\mu^{\frac{1}{2}} \left| \partial_\xi^\gamma (p^- - q^-)(x,\tau,\xi') \right| + \left| \partial_x^\gamma (p^- - q^-)(x,\tau,\xi') \right| \lesssim c_0 \mu^{-\frac{1}{2}}.$$

Taking μ sufficiently large, it will suffice to prove the bound for curves satisfying (2.31) with p^- replaced by q^- and initial data lying in the zero set of $\xi_1 + p^-$. These curves will satisfy $|\xi_1(r) + q^-(x(r), \tau(r), \xi'(r))| \lesssim \mu^{-1}$. Observe that differentiating both sides of (2.24) and dividing by $\mathrm{g}^{11}(\xi_1 - q^+(x,\tau,\xi'))$ yields

$$\partial_{x_n} q^-(x,\tau,\xi) = \frac{\partial_{x_n} \mathrm{g}^{ij} \xi_i \xi_j}{\mathrm{g}^{11}(\xi_1 - q^+(x,\tau,\xi))} + E_1.$$

where $|E_1| \lesssim c_0 \mu^{-1}$. Integrating the equation for $d\xi_n/dr$ and applying Proposition 2.1 thus gives (2.32).

For (2.33), observe that reasoning as before, we have that for some $|E_2| \lesssim c_0 \mu^{-1}$

$$\frac{dx_n}{dr} = \frac{2\xi_n(r)}{\mathrm{g}^{11}(x(r))(\xi_1(r) - q^+(r))} + E_2,$$

where we abbreviate $q^+(r) = q^+(x(r), \tau(r), \xi(r))$. Hence

$$\begin{aligned} \frac{dx_n}{dr} - \frac{\xi_n(0)}{\mathrm{g}^{11}(x(0))(\xi_1(0) - q^+(0))} &= \frac{\xi_n(r) - \xi_n(0)}{\mathrm{g}^{11}(x(r))(\xi_1(r) - q^+(r))} \\ &+ \xi_n(0) \left(\frac{1}{\mathrm{g}^{11}(x(r))(\xi_1(r) - q^+(r))} - \frac{1}{\mathrm{g}^{11}(x(0))(\xi_1(0) - q^+(0))} \right) + E_2 \end{aligned} \tag{2.34}$$

Choosing ε small in (2.14), the equations for $d\xi/dr$, dx/dr along with the bound $|\partial_{x_j} p^-| \lesssim \mu^{-\beta} c_0$ give

$$|\xi(r) - \xi(0)| \lesssim \mu^{-\beta} c_0 |r|, \qquad |x(r) - x(0)| \lesssim |r|.$$

Using that $\mathrm{g}^{11}(x)(\xi_1 - q^+)$ defines a Lipschitz function, the bound (2.33) follows. ☐

3. General Submanifolds

In this section, we prove (2.27) and hence Theorem 1.1. It suffices to treat the term u_μ^- as bounds on the u_μ^+ will follow from time reversal. Hence we suppress the superscripts on u_μ^-, G_μ^-, p^- below and assume the minus sign is taken when referencing (2.23). Recall that coordinates are chosen so that P is identified with $(y,0) \in \mathbb{R}^n$ with $y \in \mathbb{R}^k$, $0 \in \mathbb{R}^{n-k}$. In this section and the next, we take the following notational conventions on coordinates in $\mathbb{R}^n$. The letters w, y, z will denote vectors in $\mathbb{R}^k$, and given such a vector we let $\bar{y}$ denote the vector in $\mathbb{R}^n$ determined by $\bar{y} = (y,0)$. The letters x, ξ, v will typically denote vectors in $\mathbb{R}^n$ and we will often decompose such a vector as $x = (x_1, x', x'')$ where $x' = (x_2, \dots, x_k)$, $x'' = (x_{k+1}, \dots, x_n)$.

It is convenient to change the roles of t and x_1 above, and correspondingly τ and ξ_1, treating (2.25) as an equation which is hyperbolic in t, rather than in x_1. As a consequence of (2.20) and (2.21), p is now a function of (t,x,ξ) (or more precisely (t,x',x'',ξ)) satisfying the bounds

$$\left|\partial_{x,t}^\gamma \partial_\xi^\beta \left(p(t,x,\xi) - \sqrt{\xi_1^2 - |\xi'|^2}\right)\right| \lesssim c_0, \qquad |\gamma| \le 2, \tag{3.1}$$

$$\left|\partial_{x,t}^\gamma p(t,x,\xi)\right| \lesssim \mu^{\frac{1}{2}(|\gamma|-2)}, \qquad |\gamma| \ge 2, \tag{3.2}$$

for $|\xi| = 1$ in a cone of the form

$$\{\xi : -\xi_1 \gtrsim \varepsilon^{-1}|(\xi', \xi'')|\}. \tag{3.3}$$

We claim it suffices to prove the following bound over the time interval $(0,1)$

$$\|u_\mu\|_{L^q_{t,y'}L^2_{y_1}} \lesssim \mu^\delta \left(\|u_\mu\|_{L^\infty_t L^2_x} + \|G_\mu\|_{L^2_{t,x}}\right)$$

where we understand the left hand side to be

$$\left(\int_0^1 \int_{\mathbb{R}^{k-1}\times\{0\}} \left(\int_{\mathbb{R}} |u_\mu(t,\bar{y})|^2\, dy_1\right)^{\frac{q}{2}} dy'\, dt\right)^{\frac{1}{q}}, \qquad y' = (y_2, \dots, y_k)$$

and the $L^\infty_t L^2_x$ norm on right hand side as $L^\infty((0,1); L^2(\mathbb{R}^n))$. Indeed, given (2.26) we may restrict attention to estimates over the larger slab $(t,x) \in (-1,1) \times \mathbb{R}^k$ and by time translation we are reduced to proving the bound over time interval $(0,1)$. Moreover, since $p(t,x,D) - p^*(t,x,D) \in OPS^0_{1,\frac{1}{2}}$, we may differentiate $\|u_\mu(t,\cdot)\|^2_{L^2_y}$ in t to obtain

$$\|u_\mu\|_{L^\infty_t L^2_y} \lesssim \|u_\mu\|_{L^2(\mathbb{R}^{n+1})} + \|G_\mu\|_{L^2(\mathbb{R}^{n+1})}.$$

Let the wave packet transform $T_\mu : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n})$ be defined by

$$T_\mu f(x,\xi) = \mu^{\frac{n}{4}} \int e^{-i\langle \xi, v-x\rangle} \phi(\mu^{\frac{1}{2}}(v-x)) f(v)\, dv$$

where ϕ is a real valued, radial Schwartz function such that $\operatorname{supp}(\widehat{\phi})$ is contained in the unit ball and normalized so that $\|\phi\|_{L^2} = (2\pi)^{-\frac{n}{2}}$. The normalization ensures that $T_\mu^* T_\mu$ is the identity on $L^2(\mathbb{R}^n)$ and hence $\|T_\mu f\|_{L^2(\mathbb{R}^{2n}_{x,\xi})} = \|f\|_{L^2(\mathbb{R}^n_z)}$. Let $g_\mu(x) := u_\mu(0,x)$ and $\Theta_{r,t}(x,\xi)$ denote the time r value of the integral curve of

determined by the Hamiltonian flow of p with $\Theta_{r,t}(x,\xi)|_{r=t} = (x,\xi)$. Given [18, Lemma 3.2, Lemma 3.3], we may write

$$(T_\mu u_\mu)(t,x,\xi) = T_\mu g_\mu(\Theta_{0,t}(x,\xi)) + \int_0^t \tilde{G}_\mu(r,\Theta_{r,t}(x,\xi))\,dr \tag{3.4}$$

where $\tilde{G}$ satisfies

$$\int_0^t \|\tilde{G}_\mu(r,\cdot)\|_{L^2(\mathbb{R}^{2n}_{x,\xi})}\,dr \lesssim \|u_\mu\|_{L^\infty_t L^2_v} + \int_0^t \|G_\mu(r,\cdot)\|_{L^2(\mathbb{R}^n_v)}\,dr, \tag{3.5}$$

for $t \in (0,1)$. Indeed, these lemmas show that if H_p denotes the Hamiltonian vector field defined by p then $T_\mu p(\cdot,D) - H_p T_\mu$ defines an operator which is bounded on L^2 and that (3.4) follows by solving the corresponding transport equation. Furthermore, given the frequency localization of $p(\cdot,\xi)$ and the compact support of ϕ, we may assume that uniformly in r, x, we have

$$\operatorname{supp}((T_\mu g_\mu)(x,\cdot)),\ \operatorname{supp}(\tilde{G}(r,x,\cdot)) \subset \{\xi : |\xi| \approx \mu, -\xi_1 \gtrsim \varepsilon^{-1}|(\xi',\xi'')|\}. \tag{3.6}$$

Define the propagator

$$W\tilde{f}(t,y) = T^*_\mu(\tilde{f}\circ\Theta_{0,t})(\bar{y}),$$

and observe that given (3.4), (3.5) it suffices to show that

$$\|W\tilde{f}\|_{L^q_{t,y'}L^2_{y_1}} \lesssim \mu^\delta \|\tilde{f}\|_{L^2_{x,\xi}} \tag{3.7}$$

with a logarithmic adjustment for $(k,q) = (n-2,2)$. Let W_t denote the restricted operator $W_t\tilde{f}(y) = W\tilde{f}(r,y)|_{r=t}$. By duality, it suffices to show that for functions $F(s,z)$

$$\|WW^*F\|_{L^q_{t,y'}L^2_{y_1}} \lesssim \mu^{2\delta}\|F\|_{L^{q'}_{s,z'}L^2_{z_1}}. \tag{3.8}$$

To prove this, we will show that

$$\|W_tW_s^*h\|_{L^\infty_{y'}L^2_{y_1}} \lesssim \mu^{n-1}(1+\mu|t-s|)^{-\frac{n-1}{2}}\|h\|_{L^1_{y'}L^2_{y_1}} \tag{3.9}$$

$$\|W_tW_s^*h\|_{L^2_y} \lesssim \mu^{n-k}(1+\mu|t-s|)^{-\frac{n-k}{2}}\|h\|_{L^2_y} \tag{3.10}$$

When $k = n-2$ and $q = 2$, Young's inequality and (3.10) give (3.8) with the logarithmic loss. In all other cases with $k \le n-2$, we may interpolate (3.9) and (3.10) to obtain

$$\|W_tW_s^*h\|_{L^q_{y'}L^2_{y_1}} \lesssim \mu^{2(\frac{n-1}{2}-\frac{k-1}{q})}(1+\mu|t-s|)^{-(\frac{n-1}{2}-\frac{k-1}{q})}\|h\|_{L^{q'}_{y'}L^2_{y_1}} \tag{3.11}$$

and use that $(1+|s|)^{-(\frac{n-1}{2}-\frac{k-1}{q})} \in L^{q/2}(\mathbb{R})$ to get (3.8). The same argument works when $k = n-1$ and $\frac{2n}{n-1} < q < \infty$. To handle the remaining cases when $k = n-1$, we use that

$$\mu^{2(\frac{n-1}{2}-\frac{n-2}{q})}(1+\mu|t-s|)^{-(\frac{n-1}{2}-\frac{n-2}{q})} \lesssim \mu^{\frac{n-1}{2}-\frac{n-2}{q}}|t-s|^{-\frac{n-1}{2}+\frac{n-2}{q}}.$$

Hence (3.8) follows from the Hardy-Littlewood-Sobolev inequality when $q = \frac{2n}{n-2}$. When $2 \le q < \frac{2n}{n-2}$, the right hand side is in $L^{q/2}_{loc}$ and Young's inequality gives (3.8).

The bound (3.9) is well-established as it is crucial for $L^q(\mathbb{R}^n)$ bounds on the squarefunction, see Smith [18, (3.5)] and Smith-Sogge [17, (5.4), (7.2)]. It thus

suffices to prove the second. Using that $(x,\xi) \mapsto \Theta_{r_1,r_2}(x,\xi)$ defines diffeomorphism which preserves $dx \wedge d\xi$, the kernel of $W_t W_s^*$ can be realized as (cf. [17, p.127])

$$K_{t,s}(y,z) = \mu^{\frac{n}{2}} \int e^{i\langle \xi, \bar{y}-x\rangle - i\langle \xi_{s,t}, \bar{z}-x_{s,t}\rangle} \phi(\mu^{\frac{1}{2}}(\bar{y}-x))\phi(\mu^{\frac{1}{2}}(\bar{z}-x_{s,t}))\Gamma(\xi)\, dx d\xi \tag{3.12}$$

with $(x_{s,t}, \xi_{s,t})$ abbreviating $(x_{s,t}(x,\xi), \xi_{s,t}(x,\xi))$. Here Γ is a cutoff supported in a region of the form appearing in (3.6) which may be inserted since we are only interested in functions $\tilde{f}$ satisfying that condition.

Before proceeding further, we observe bounds on the bicharacteristic flow of p.

Theorem 3.1. *Suppose $(x,\xi) \in \mathbb{R}^{2n}$ with ξ in the set defined by* (3.3)*. Let $\Theta_{s,t}(x,\xi)$ be as in* (3.4)*, that is, $\Theta_{s,t}(x,\xi)|_{s=t} = (x,\xi)$ and*

$$\partial_s x_{s,t}(x,\xi) = d_\xi p(s, \Theta_{s,t}(x,\xi)), \qquad \partial_s \xi_{s,t}(x,\xi) = -d_x p(s, \Theta_{s,t}(x,\xi)). \tag{3.13}$$

Then for $s,t \in [0,1]$, first partials of $x_{s,t}(x,\xi)$, $\xi_{s,t}(x,\xi)$ in x,ξ satisfy

$$|d_x x_{s,t} - I| + |d_x \xi_{s,t}| \lesssim c_0|t-s|, \tag{3.14}$$

$$\left| d_\xi x_{s,t}(x,\xi) - \int_t^s d_\xi d_\xi p(\Theta_{r,t}(x,\xi))\, dr \right| + |d_\xi \xi_{s,t}(x,\xi) - I| \lesssim c_0|t-s|^2. \tag{3.15}$$

Proof. Differentiating the equations (3.13) gives

$$\partial_r \begin{bmatrix} dx_{r,t} \\ d\xi_{r,t} \end{bmatrix} = M(r, x_{r,t}, \xi_{r,t}) \begin{bmatrix} dx_{r,t} \\ d\xi_{r,t} \end{bmatrix}, \qquad \text{where } M = \begin{bmatrix} d_x d_\xi p & d_\xi d_\xi p \\ -d_\xi d_x p & d_x d_x p \end{bmatrix}.$$

By Gronwall's inequality and the bounds (3.1) we have

$$|d_x x_{r,t} - I| + |d_x \xi_{r,t}| \lesssim 1, \qquad |d_\xi x_{r,t}| + |d_\xi \xi_{r,t} - I| \lesssim 1,$$

and substituting these bounds back into the integral equation for $dx_{r,t}, d\xi_{r,t}$ implies the theorem. $\square$

This type of argument can also be used to bound higher order derivatives of $x_{r,t}, \xi_{r,t}$, see e.g. [17, Theorem 5.1]. Such bounds are used in the proof of the next theorem. We omit them since we rely on [17] for the proof.

The next result is due to Smith-Sogge (see [17, Theorem 5.4] for (3.17) and [17, p.152] for (3.18)) which obtains bounds on $K_{t,s}$ under the assumption that Γ is a smooth cutoff to a (possibly) smaller set. The bounds (3.17) will play a crucial role in the proof of (3.10) in this section and (3.18) will be used in §4.

Theorem 3.2. *Suppose $\bar{\theta} = \max(1, \mu^{-\frac{1}{2}}|t-s|^{-\frac{1}{2}})$ and the smooth cutoff Γ in* (3.12) *is supported in a set contained in* (3.6) *of the form*

$$supp(\Gamma) \subset \{\xi : |\xi/|\xi| - \eta| \lesssim \bar{\theta}\} \tag{3.16}$$

for some unit vector $\eta \in \mathbb{S}^{n-1}$. Let $(x_{s,t}, \nu_{s,t}) = \Theta_{s,t}(\bar{z}, \eta)$. Then $K_{t,s}$ satisfies the pointwise bounds

$$|K_{t,s}(y,z)| \lesssim \mu^n \bar{\theta}^{n-1}(1 + \mu\bar{\theta}|\bar{y} - x_{s,t}| + \mu|\langle \nu_{s,t}, \bar{y} - x_{s,t}\rangle|)^{-N}. \tag{3.17}$$

Moreover, if $k = n-1$ (so that $\xi'' = \xi_n$ is one dimensional) and Γ is a smooth cutoff to set of the form

$$supp(\Gamma) \subset \left\{ \xi : |\xi_n - r| \lesssim \mu\theta, \left| \frac{(\xi_1, \xi')}{|(\xi_1, \xi')|} - \eta \right| \lesssim \bar{\theta} \right\},$$

where this time $\eta \in \mathbb{S}^{n-2}$, $\theta \le \bar\theta$ *and* $|r| \lesssim \mu\theta$, *we have*

$$|K_{t,s}(y,z)| \lesssim \mu^n \bar\theta^{n-2}\theta(1+\mu\bar\theta|y'-x'_{s,t}|+\mu|\langle \nu_{s,t},\bar y - x_{s,t}\rangle|)^{-N}. \tag{3.18}$$

Observing that $\mu^{n-k}(1+\mu|t-s|)^{-\frac{n-k}{2}} \approx \min(\mu^{n-k},\mu^{\frac{n-k}{2}}|t-s|^{-\frac{n-k}{2}})$, we begin treating the case $|t-s| \le \mu^{-1}$, that is, the case where the first quantity is smaller. In this case, we apply (3.17) in Theorem 3.2 with $\bar\theta = 1$ and $\eta = -e_1$ to obtain

$$|K_{t,s}(y,z)| \lesssim \mu^n(1+\mu|\bar y - x_{s,t}(z,-e_1)|)^{-N}$$

which gives the first half of (3.19) below. Making the measure preserving change of variables $(x,\xi) \mapsto (x_{t,s}(x,\xi),\xi_{t,s}(x,\xi))$ in (3.12), we may reverse the roles of y and z in Theorem 3.2 to obtain an analogous bound which yields

$$\int |K_{t,s}(y,z)|\,dy + \int |K_{t,s}(y,z)|\,dz \lesssim \mu^{n-k} \tag{3.19}$$

(strictly speaking, the change of variables replaces $\Gamma(\xi)$ by $\Gamma(\xi_{s,t}(x,\xi))$, but this does not change the validity of the bounds in Theorem 3.2).

It now suffices to treat the more involved case where $\mu^{-1} < |t-s| \le 1$, and for the remainder of this section we assume $t,s \in [0,1]$ are two fixed values satisfying this condition. Using the notation suggested by Theorem 3.2, we set $\bar\theta = \mu^{-\frac12}|t-s|^{-\frac12}$ so that $\mu\bar\theta^2|t-s| = 1$. Using a partition of unity, we take a decomposition $K_{t,s} = \sum_j K^j$ where K^j is defined by replacing Γ in (3.12) by a smooth cutoff Γ_j, with Γ_j supported in a set of the form $|\xi/|\xi| - \eta^j| \lesssim \bar\theta$ and $\{\eta^j\}$ is a collection of unit vectors in the intersection of $\mathbb{S}^{n-1}$ with the cone $\{-\xi_1 \gtrsim \varepsilon^{-1}|(\xi',\xi'')|\}$ separated by a distance of at least $\approx \bar\theta^{-1}$. In particular, we may assume that for fixed j

$$\sum_l (1+\bar\theta^{-1}|\eta^j-\eta^l|)^{-(n+1)} \lesssim 1. \tag{3.20}$$

Let T_j be the operator defined by $(T_jh)(y) = \int K^j(y,z)h(z)\,dz$ and observe that since $|(\nu^j)_1| \approx 1$, (3.17) in Theorem 3.2 with $\eta = \eta^j$ gives

$$\int |K^j(y,z)|\,dy \lesssim \mu^{n-k}\bar\theta^{n-k}.$$

By the same symmetry argument used in (3.19), we now have

$$\|T_jh\|_{L^2_y} \lesssim \mu^{n-k}\bar\theta^{n-k}\|h\|_{L^2_y} = \mu^{\frac{n-k}{2}}|t-s|^{-\frac{n-k}{2}}\|h\|_{L^2_y}$$

(though in what follows, it is convenient to express the bounds in terms of μ, $\bar\theta$).

We claim that there exists a constant C such that if $\bar\theta^{-1}|\eta^j-\eta^l| \ge C$, then

$$\|T_l^*T_j\|_{L^2\to L^2} + \|T_lT_j^*\|_{L^2\to L^2} \lesssim \mu^{2(n-k)}\bar\theta^{2(n-k)}(1+\bar\theta^{-1}|\eta^j-\eta^l|)^{-N}.$$

Since $W_tW_s^* = \sum_j T_j$, Cotlar's lemma then implies (3.10). Furthermore, we focus on the bound for $T_l^*T_j$ as $(T_lT_j^*)^* = T_jT_l^*$ and the weights below are symmetric in j and l. To prove this inequality, we set

$$J_{j,l}(z,w) = \int \overline{K^l(y,z)}K^j(y,w)\,dy$$

and will show that for $\bar\theta^{-1}|\eta^j-\eta^l| \ge C$,

$$|J_{j,l}(z,w)| \lesssim \mu^{2n-k}\bar\theta^{2n-1-k}(1+\mu\bar\theta|z-w|+\mu|\langle\eta^l,\bar z-\bar w\rangle|+\bar\theta^{-1}|\eta^j-\eta^l|)^{-N}. \tag{3.21}$$

The proof of (3.21) varies based on whether $|(\eta_1^j - \eta_1^l, \dots, \eta_k^j - \eta_k^l)| \geq |(\eta^j - \eta^l)''|$ or the opposite inequality holds. In the first case, we write

$$(3.22)\quad J_{j,l}(z,w) = \mu^{\frac{n}{2}} \int\int \left(\int e^{i\langle \xi, \bar{y}-x\rangle - i\langle \tilde{\xi}, \bar{y}-\tilde{x}\rangle} \phi(\mu^{\frac{1}{2}}(\bar{y}-x))\phi(\mu^{\frac{1}{2}}(\bar{y}-\tilde{x}))\, dy \right) \times \psi(z,w,x,\xi,\tilde{x},\tilde{\xi})\Gamma_j(\xi)\Gamma_l(\tilde{\xi})\, dx d\xi\, d\tilde{x} d\tilde{\xi}$$

where $(\tilde{x}, \tilde{\xi})$ denote the variables in the integral defining K_l and ψ is a function independent of y. The y integral in parentheses is a constant multiple of

$$(3.23)\qquad \int e^{i\tilde{\psi}}\, \widehat{\phi}(\mu^{-\frac{1}{2}}((\zeta_1,\zeta',\zeta'')-\xi))\widehat{\phi}(\mu^{-\frac{1}{2}}((\zeta_1,\zeta',\tilde{\zeta}'')-\tilde{\xi}))\, d\zeta_1 d\zeta' d\zeta'' d\tilde{\zeta}''$$

where $\tilde{\psi}$ is some real valued phase function. Since $\mathrm{supp}(\widehat{\phi})$ is contained in the unit ball and $2|(\eta_1^j - \eta_1^l, \dots, \eta_k^j - \eta_k^l)| \geq |\eta^j - \eta^l|$, this integral vanishes if $\bar{\theta}^{-1}|\eta^l - \eta^j| \geq C$ as this implies that $|(\xi_1 - \tilde{\xi}_1, \dots, \xi_k - \tilde{\xi}_k)| \gtrsim C\mu\bar{\theta} \geq C\mu^{\frac{1}{2}}$.

We now turn to the case where $|(\eta^j)'' - (\eta^l)''| \geq |(\eta_1^j - \eta_1^l, \dots, \eta_k^j - \eta_k^l)|$. In this case, we use (3.17) in Theorem 3.2 to bound $|K_l|$, $|K_j|$ individually. After some minor manipulations, this yields

$$(3.24)\quad |J_{j,l}(z,w)| \lesssim \mu^{2n}\bar{\theta}^{2(n-1)} \times \int (1 + \mu\bar{\theta}|\bar{y} - x_{s,t}(\bar{w},\eta^j)| + \mu\bar{\theta}|\bar{y} - x_{s,t}(\bar{z},\eta^l)| + \mu|\langle \nu_{s,t}(\bar{z},\eta^l), \bar{y} - x_{s,t}(\bar{z},\eta^l)\rangle|)^{-6N} \times (1 + \mu|\langle \nu_{s,t}(\bar{w},\eta^j), \bar{y} - x_{s,t}(\bar{w},\eta^j)\rangle - \langle \nu_{s,t}(\bar{z},\eta^l), \bar{y} - x_{s,t}(\bar{z},\eta^l)\rangle|)^{-N}\, dy$$

We take $3N$ of the powers in the first factor of the integrand on the right and claim that up to implicit constants, it is bounded above by

$$(3.25)\qquad (1 + \mu\bar{\theta}|z-w| + \bar{\theta}^{-1}|\eta^j - \eta^l|)^{-3N}$$

To see this, first observe that the $3N$ powers from the integrand are dominated by

$$(1 + \mu\bar{\theta}|x_{s,t}(\bar{z},\eta^l) - x_{s,t}(\bar{w},\eta^j)| + 64\mu\bar{\theta}|x''_{s,t}(\bar{z},\eta^l) - x''_{s,t}(\bar{w},\eta^j)|)^{-3N}.$$

By the bounds (3.14), (3.15) in Theorem 3.1, we have

$$(3.26)\qquad |x_{s,t}(\bar{z},\eta^l) - x_{s,t}(\bar{w},\eta^j)| \geq \frac{3}{4}|z-w| - 2|t-s||\eta^l - \eta^j|$$

provided c_0 and ε are taken sufficiently small. Next we use that

$$\left|x''_{s,t}(\bar{w},\eta^j) - x''_{s,t}(\bar{z},\eta^l)\right| \geq \left|x''_{s,t}(\bar{w},\eta^j) - x''_{s,t}(\bar{w},\eta^l)\right| - \left|x''_{s,t}(\bar{w},\eta^l) - x''_{s,t}(\bar{z},\eta^l)\right|.$$

To bound the second term on the right, we use that as a consequence of (3.14) the $(n-k)\times n$ matrix $d_x x''_{s,t}$ satisfies

$$\left|d_x x''_{s,t} - [0\ \ I_{n-k}]\right| \lesssim c_0|t-s|.$$

Recalling that $\bar{w} = (w,0)$, $\bar{z} = (z,0)$, this gives

$$\left|x''_{s,t}(\bar{w},\eta^l) - x''_{s,t}(\bar{z},\eta^l)\right| \lesssim c_0|t-s||z-w|.$$

We now use (3.1), (3.15) to get that $d_\xi x''_{s,t}(x,\xi)$ is the $(n-k)\times n$ block matrix

$$(s-t)(\xi_1^2 - |(\xi',\xi'')|^2)^{-3/2}\left[\xi_1\xi'' \quad -\xi''(\xi')^T \quad -\Big((\xi_1^2 - |(\xi',\xi'')|^2)I_{n-k} + \xi''(\xi'')^T\Big)\right]$$

plus an error term which is $\mathcal{O}(c_0|t-s|)$. Here ξ'' is taken to be a column vector. Using that $|(\eta^l-\eta^j)''| \geq |\eta^l-\eta^j|/2$ and $|(\xi',\xi'')| \lesssim \varepsilon|\xi_1|$, we have that

$$\left|x''_{s,t}(\bar{w},\eta^j)-x''_{s,t}(\bar{w},\eta^l)\right| \geq \frac{|t-s|}{8}|\eta^l-\eta^j|.$$

In summary, we have that for some uniform constant M,

$$64\left|x''_{s,t}(\bar{w},\eta^j)-x''_{s,t}(\bar{z},\eta^l)\right| \geq 8|t-s||\eta^l-\eta^j| - Mc_0|t-s||z-w| \tag{3.27}$$

By taking c_0 sufficiently small, the negative term in (3.27) can be absorbed by the first term in (3.26) and vice versa, which shows (3.25).

We now turn to the second factor in the integrand of (3.24). The triangle inequality gives

$$\mu\left|\langle \nu_{s,t}(\bar{w},\eta^j), \bar{y}-x_{s,t}(\bar{w},\eta^j)\rangle - \langle \nu_{s,t}(\bar{z},\eta^l), \bar{y}-x_{s,t}(\bar{z},\eta^l)\rangle\right| \geq \mu|\langle \eta^j, \bar{z}-\bar{w}\rangle| - R$$

with

$$\begin{aligned} R = \mu\left|\nu_{s,t}(\bar{z},\eta^l)-\nu_{s,t}(\bar{w},\eta^j)\right|\left|\bar{y}-x_{s,t}(\bar{z},\eta^l)\right| \\ + \mu\left|\langle \nu_{s,t}(\bar{w},\eta^j), x_{s,t}(\bar{z},\eta^l)-x_{s,t}(\bar{w},\eta^j)\rangle - \langle \eta^j,\bar{z}-\bar{w}\rangle\right|. \end{aligned}$$

We claim that

$$R \lesssim (\mu\bar{\theta})^2\left|\bar{y}-x_{s,t}(\bar{z},\eta^l)\right|^2 + \bar{\theta}^{-2}|\eta^j-\eta^l|^2 + (\mu\bar{\theta})^2|z-w|^2 + 1. \tag{3.28}$$

The error induced by R can thus be absorbed by $2N$ of the powers in (3.25) and $2N$ of the powers in the first factor in (3.24). This will conclude the proof of (3.21) since the remaining N powers of the first factor in (3.24) can be used to integrate in y.

To bound the first term in R, we use the geometric-arithmetic mean inequality and observe that the bounds on $d_x\xi_{s,t}$, $d_\xi\xi_{s,t}$ in Theorem 3.1 give

$$\bar{\theta}^{-1}\left|\nu_{s,t}(\bar{z},\eta^l)-\nu_{s,t}(\bar{w},\eta^j)\right| \lesssim \bar{\theta}^{-1}|z-w| + \bar{\theta}^{-1}|\eta^j-\eta^l|.$$

This is seen to be bounded by the right hand side of (3.28) after recalling that $\bar{\theta}^{-1} \leq \mu\bar{\theta}$ when $|t-s| \leq 1$. Since $\mu\bar{\theta}^2|s-t| = 1$ and $\mu \leq (\mu\bar{\theta})^2$ the rest of (3.28) follows from

$$\left|\langle \nu_{s,t}(\bar{w},\eta^j), x_{s,t}(\bar{z},\eta^l)-x_{s,t}(\bar{w},\eta^j)\rangle - \langle \eta^j,\bar{z}-\bar{w}\rangle\right| \lesssim |z-w|^2 + \bar{\theta}^2|s-t|$$

which can be seen by differentiating the expression on the left in s see [17, p.133].

4. Curved Submanifolds

In this section, we prove the bound (2.30) and hence Theorem 1.2. Recall the reductions in §2.1, in particular that Fermi coordinates are taken. Also, we take the same notational conventions as in the beginning of the previous section, though it is convenient to replace x'' by x_n, so that $x \in \mathbb{R}^n$ is decomposed as $x = (x_1, x', x_n)$. The wave packet transform from above can also be used here, and after following the initial reductions in §3, it suffices to show that the propagator

$$\begin{aligned} W\tilde{f}(t,y) &= T^*_\mu(\tilde{f}\circ\Theta_{0,t})(\bar{y}) \\ &= \mu^{\frac{n}{4}}\int e^{i\langle \xi_{t,0}(x,\xi),\bar{y}-x_{t,0}(x,\xi)\rangle}\phi(\mu^{\frac12}(\bar{y}-x_{t,0}(x,\xi)))\tilde{f}(x,\xi)\,dxd\xi \end{aligned}$$

satisfies

$$\|W\tilde{f}\|_{L^2_{t,y}} \lesssim \mu^{\frac16(1+\beta)}\|\tilde{f}\|_{L^2_{x,\xi}}, \qquad \beta = \frac{\sigma}{1-\sigma} < \frac12. \tag{4.1}$$

where $\tilde{f}$ is a function supported in a region of the form (3.6). However, given (2.26), we may assume (t, y) are restricted to $(0,1) \times (-1,1)^{n-1}$, that is, we bound $L^2((0,1) \times (-1,1)^{n-1})$ norm of $W\tilde{f}$. This will allow us apply Theorem 2.2 below.

Let N_μ, n_μ be integers such that $N_\mu \approx \log_2(\mu^{\frac{1}{3}(1+\beta)})$, $n_\mu \approx \log_2(\mu^\beta)$ and take a smooth partition of unity $\{\Gamma_j(\xi)\}_{j=n_\mu}^{N_\mu}$ satisfying $\sum_j \Gamma_j \equiv 1$ on $\mathrm{supp}(\tilde{f})$ and

$$\begin{aligned} \mathrm{supp}(\Gamma_{n_\mu}) &\subset \{\xi : |\xi_n| \geq \mu 2^{-n_\mu - 2}\}, \\ \mathrm{supp}(\Gamma_j) &\subset \{\xi : |\xi_n| \in [\mu 2^{-j-2}, \mu 2^{-j+2}]\}, \qquad n_\mu < j < N_\mu \\ \mathrm{supp}(\Gamma_{N_\mu}) &\subset \{\xi : \xi_n \in [-\mu 2^{-N_\mu+2}, \mu 2^{-N_\mu+2}]\} \end{aligned}$$

For each $n_\mu \leq j \leq N_\mu$, we define

$$W^j\tilde{f}(t,y) = \mu^{\frac{n}{4}} \int e^{i\langle \xi_{t,0}(x,\xi), \bar{y} - x_{t,0}(x,\xi)\rangle} \phi(\mu^{\frac{1}{2}}(\bar{y} - x_{t,0}(x,\xi)))\Gamma_j(\xi_{t,0}(x,\xi))\tilde{f}(x,\xi)\, dx d\xi$$

and observe that it suffices to show that

$$\|W^j\tilde{f}\|_{L^2_{t,y}} \lesssim 2^{\frac{j}{2}}\|\tilde{f}\|_{L^2_{x,\xi}}. \tag{4.2}$$

As before, we let $W^j_t\tilde{f}(y) = W^j\tilde{f}(r,y)|_{r=t}$.

We begin with the case $j = N_\mu$ and claim there exists $\tilde{c}_1$ such that

$$W_t^{N_\mu}(W_s^{N_\mu})^* = 0 \qquad \text{whenever} \qquad |t-s| \geq \tilde{c}_1\mu^\beta 2^{-N_\mu}. \tag{4.3}$$

To see this, observe that after a change of variable, the kernel of $W_t^{N_\mu}(W_s^{N_\mu})^*$ is

$$\mu^{\frac{n}{2}} \iint e^{i\langle \xi, \bar{y}-x\rangle - i\langle \xi_{s,t}, \bar{z} - x_{s,t}\rangle}\phi(\mu^{\frac{1}{2}}(\bar{y}-x))\phi(\mu^{\frac{1}{2}}(\bar{z}-x_{s,t}))\Gamma_{N_\mu}(\xi)\Gamma_{N_\mu}(\xi_{s,t})\, dx d\xi. \tag{4.4}$$

Applying (2.32) in Theorem 2.2, there exists a constant $\tilde{c}_1$, inversely proportional to c_1 above, such that $|(\xi_{t,s})_n(x,\xi)| \geq \mu 2^{-N_\mu+2}$ whenever $\mu^{-\beta}|t-s| \geq \tilde{c}_1 2^{-N_\mu}$ and $\xi \in \mathrm{supp}(\Gamma_{N_\mu})$.

For $|t-s| \leq \tilde{c}_1\mu^\beta 2^{-N_\mu}$, we write the kernel in (4.4) as $\sum_i K_i(y,z)$ where K_i is defined by replacing $\Gamma_{N_\mu}(\xi)$ in (4.4) with a smooth cutoff to a region of the form

$$\{\xi : -\xi_1 \approx \mu, |\xi_n| \lesssim \mu 2^{-N_\mu}, \left|(\xi_1,\xi')/|(\xi_1,\xi')| - \eta^i\right| \lesssim \bar{\theta}\}, \quad \bar{\theta} = \max(1, \mu^{-\frac{1}{2}}|t-s|^{-\frac{1}{2}})$$

where this time η^i is a collection of vectors on $\mathbb{S}^{n-2}$ mutually separated by a distance $\approx \bar{\theta}$ so that (3.20) holds. By Cotlar's lemma and a slight adjustment of the almost orthogonality argument in (3.23), it suffices to show

$$\int |K_i(y,z)|\, dy + \int |K_i(y,z)|\, dz \lesssim \mu 2^{-N_\mu}. \tag{4.5}$$

Indeed, if this bound holds then by Young's inequality in t, s we have

$$\|W^{N_\mu}\left(W^{N_\mu}\right)^*\|_{L^2_{s,z}\to L^2_{t,y}} \lesssim \mu 2^{-N_\mu}\cdot \mu^\beta 2^{-N_\mu} \approx 2^{N_\mu} \tag{4.6}$$

and (4.2) follows by duality. But the first half of (4.5) is a consequence of (3.18) in Theorem 3.2 with $\eta = \eta^i$ and $\theta = 2^{-N_\mu}$ and the second half follows by symmetry and the same theorem. Indeed (3.18) applies here as our assumption on $|t-s|$ means that $\bar{\theta} \gtrsim \mu^{-\frac{1}{2}(1+\beta)}2^{\frac{1}{2}N_\mu} \approx 2^{-N_\mu}$.

For $n_\mu \leq j < N_\mu$, we take a partition of unity over $\mathbb{R}^{n-1}$, $\sum_l \chi(y-l) \equiv 1$ such that the sum is taken over $l \in \mathbb{Z}^{n-1}$ and $\mathrm{supp}(\chi) \subset [-1,1]^{n-1}$. Use this to define

$$\chi_l(y) := \chi(\mu^{-\beta}2^j y - l) \quad \text{and} \quad W^{j,l}\tilde{f}(t,y) := \chi_l(y)W^j\tilde{f}(t,y)$$

and we consider only those l such that $\operatorname{supp}(\chi_l)$ intersects $(-1,1)^n$. By the support properties of χ we may take C sufficiently large so that $(W^{j,m})^* W^{j,l}$ vanishes whenever $|l-m| \geq C$. We next claim that we can take C so that

$$\|W^{j,l}(W^{j,m})^*\|_{L^2\to L^2} \lesssim \mu^{-N} \qquad \text{whenever } |l-m| \geq C. \tag{4.7}$$

Since there is at most $\mathcal{O}(\mu^{\frac{n-1}{3}})$ of the $W^{j,l}$, the estimate (4.2) on W^j will follow by Cotlar's lemma and Young's inequality provided we can show

$$\|W^{j,l}_t(W^{j,l}_s)^* h\|_{L^2_y} \lesssim \mu 2^{-j}(1+\mu 2^{-2j}|t-s|)^{-2}\|h\|_{L^2_y}. \tag{4.8}$$

In order to show (4.7), we write the kernel of the operator as

$$K^{l,m}_{t,s}(y,z) = \chi_l(y)\chi_m(z)\mu^{\frac{n}{2}} \times \iint e^{i\langle \xi,\bar{y}-x\rangle - i\langle \xi_{s,t},\bar{z}-x_{s,t}\rangle}\phi(\mu^{\frac12}(\bar{y}-x))\phi(\mu^{\frac12}(\bar{z}-x_{s,t}))\Gamma_j(\xi)\Gamma_j(\xi_{s,t})\,dxd\xi. \tag{4.9}$$

Given the compact support of $K^{l,m}_{t,s}$ in y and z it suffices to show that the integral is dominated by μ^{-N} for any N. Similar to the $j=N_\mu$ case, if $\xi\in\operatorname{supp}(\Gamma_j)$ and $|t-s|\geq \tilde{c}_1\mu^\beta 2^{-j}$ for some $\tilde{c}_1$ depending only on c_1, then $\Gamma_j(\xi_{t,s}(x,\xi))=0$, meaning the kernel vanishes for such t,s. When $|t-s|\leq \tilde{c}_1\mu^\beta 2^{-j}$, we use that

$$|x_{s,t}(\bar{y},\xi)-x_{s,t}(x,\xi)| \lesssim |\bar{y}-x|$$

to bound the integral in (4.9) by

$$\iint (1+\mu^{\frac12}|\bar{y}-x| + \mu^{\frac12}|\bar{z}-x_{s,t}(\bar{y},\xi)|)^{-2N}\Gamma_j(\xi)\,dxd\xi.$$

Using the elementary estimate $|x_{t,s}(\bar{y},\xi)-\bar{y}|\leq 2|t-s|$, we see that if $|l-m|\geq 2^4\tilde{c}_1$,

$$\begin{aligned}|\bar{z}-x_{s,t}(\bar{y},\xi)| &\geq |y-z| - 2|t-s| \\ &\geq \mu^\beta 2^{-j-2}|l-m| - 2\tilde{c}_1\mu^\beta 2^{-j} \geq \mu^\beta 2^{-j} \geq \mu^{-\frac13}\end{aligned}$$

and hence $\mu^{\frac12}|\bar{z}-x_{s,t}(\bar{y},\xi)| \gtrsim \mu^{\frac16}$. This implies the desired bound on (4.9).

We now turn to (4.8). It suffices to restrict attention to $|t-s|\leq\tilde{c}_1\mu^\beta 2^{-j}$, though this does not play a crucial role in the argument. First consider the case where t,s satisfy $|t-s|\leq \mu^{-1}2^{2j}$. We begin by observing that a slight adjustment of the almost orthogonality argument in (3.23) and (4.6) allows us to assume that the kernel of $W^{j,l}_t(W^{j,l}_s)^*$ takes the form

$$K^{l,l}_{t,s}(y,z) = \chi_l(y)\chi_l(z)\mu^{\frac{n}{2}} \times \iint e^{i\langle \xi,\bar{y}-x\rangle - i\langle \xi_{s,t},\bar{z}-x_{s,t}\rangle}\phi(\mu^{\frac12}(\bar{y}-x))\phi(\mu^{\frac12}(\bar{z}-x_{s,t}))\widetilde{\Gamma}_j(\xi)\Gamma_j(\xi_{s,t})\,dxd\xi. \tag{4.10}$$

with

$$\operatorname{supp}(\widetilde{\Gamma}_j) \subset \{\xi : -\xi_1\approx\mu, |\xi_n|\approx \mu 2^{-j}, |(\xi_1,\xi')/|(\xi_1,\xi')| - \eta| \lesssim \bar{\theta}\},$$

for some fixed vector $\eta\in\mathbb{S}^{n-2}$ and $\bar{\theta} = \max(1,\mu^{-\frac12}|t-s|^{-\frac12}) \geq 2^{-j}$. Indeed, reasoning as in (3.22), we are lead to consider the integral

$$\int e^{i\langle\xi,\bar{y}-x\rangle - i\langle\tilde{\xi},\bar{y}-\tilde{x}\rangle}\phi(\mu^{\frac12}(\bar{y}-x))\phi(\mu^{\frac12}(\bar{y}-\tilde{x}))\chi_l^2(y)\,dy.$$

While this integral does not vanish when $\mu^{\frac12} \ll \mu\bar{\theta} \leq |\xi-\tilde{\xi}|$, we may bound its absolute value by $C_N\mu^{-N}$ for any N, which is just as effective. Indeed, we may

take the Fourier transform similarly to (3.23) and since the Fourier transform of χ_l^2 is concentrated (though not localized) in a ball of radius $\mu^{-\beta}2^j \leq \mu^{\frac{1}{3}} \ll \mu^{\frac{1}{2}}$, the rapid decay in μ follows. We now conclude (4.8) for $|t-s| \leq \mu^{-1}2^{2j}$ by applying (3.18) with $\theta = 2^{-j}$ in Theorem 3.2 and reasoning analogously to (4.5).

To show (4.8) when $|t-s| > \mu^{-1}2^{2j}$, we take the decomposition used in §3, writing the kernel $K^{l,l}_{t,s} = \sum_i K_i$ with K_i is defined by replacing the $\widetilde{\Gamma}_j$ in (4.10) by a smooth cutoff $\widetilde{\Gamma}_{j,i}$ to a region of the form

$$\{\xi \in \mathrm{supp}(\Gamma_j) : -\xi_1 \approx \mu, |\xi/|\xi| - \eta^i| \lesssim \bar{\theta}\}, \qquad \bar{\theta} = \mu^{-\frac{1}{2}}|t-s|^{-\frac{1}{2}}$$

where $\eta_i \in \mathbb{S}^{n-1}$. As before, we assume that η^i are separated so that (3.20) holds. The estimates (3.17) in Theorem 3.2 give that

$$|K_i(t,y;s,z)| \lesssim \mu^n \bar{\theta}^{n-1}(1 + \mu\bar{\theta}|\bar{y} - x^i_{t,s}| + \mu|\langle \nu^i_{t,s}, \bar{y} - x^i_{t,s}\rangle|)^{-N} \tag{4.11}$$

We will show that

$$|\bar{y} - x^i_{t,s}| \gtrsim 2^{-j}|t-s|. \tag{4.12}$$

Together with our assumption on t,s this gives $\mu^{\frac{1}{2}}2^{-j}|t-s|^{\frac{1}{2}} \lesssim \mu\bar{\theta}|\bar{y} - x^i_{t,s}|$, and hence this additional decay and the almost orthogonality arguments above can be integrated into the proof of (3.21) to obtain

$$\|W^{j,l}_t (W^{j,l}_s)^*\|_{L^2 \to L^2} \lesssim \mu\bar{\theta}(1 + \mu 2^{-2j}|t-s|)^{-2} \leq \mu 2^{-j}(1 + \mu 2^{-2j}|t-s|)^{-2}.$$

To show (4.12), first consider t,s satisfying $\mu^{-1}2^{2j} < |t-s| \leq \mu^{\beta}2^{-j+3}$. We may assume c_0 in (2.20) is sufficiently small and linearize the n-th component of $x_{t,s}\left(\bar{z}, \eta^i\right)$ as in Theorem 2.2 to obtain

$$\left|(x_{t,s})_n\left(\bar{z}, \eta^i\right)\right| \gtrsim 2^{-j}|t-s|$$

since the n-th component of $\bar{z}$ vanishes. Indeed, over this time scale, the error term is smaller than the linearization.

Now assume that $|t-s| \geq \mu^{\beta}2^{-j+3}$. Taking ε, c_0 sufficiently small, we have that

$$\left|(x_{t,s})_1(\bar{z}, \eta^i) - z_1\right| = \left|\int_s^t \partial_{\xi_1} p(r, \Theta_{r,s}(\bar{z}, \eta^i))\, dr\right| \geq \frac{1}{2}|t-s|$$

Using that $y, z \in \mathrm{supp}(\chi_l)$ we have that $|y-z| \leq 2^{-j+1}\mu^{\beta} \leq \frac{1}{4}|t-s|$ and hence we have the stronger bound

$$\left|(x_{t,s})_1(\bar{z}, \eta^i) - y_1\right| \geq \frac{1}{2}|t-s| - |z_1 - y_1| \geq \frac{1}{4}|t-s|.$$

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